Asymptotic Properties of Linearized Equations of Low Compressible Fluid Motion*

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Abstract

Initial–boundary value problem for linearized equations of motion of viscous barotropic fluid in a bounded domain is considered. Existence, uniqueness and estimates of weak solutions to this problem are derived. Convergence of the solutions towards the incompressible limit when compressibility tends to zero is studied.

1 Introduction

In many cases mathematical treatment of liquids is done in the framework of *incompressible* fluid. However, from the physical point of view, all the liquids existing in nature are *low compressible*. Therefore it is reasonable to study asymptotic properties of solutions to equations of low compressible fluid, in particular, convergence to the corresponding incompressible limit.

Low Mach number limit, which can be considered as a particular case of low compressibility limit, was studied by E. Feireisl, P.-L. Lions and others [1, 2, 3, 4]. In particular, it was proved (see [1, 4]) that there exists a sequence of weak solutions to equations of compressible fluid motion which converges to a solution of the corresponding equations of incompressible fluid. More precisely, *weak* convergence of the velocity was established. But from the physical consideration, one could desire *strong* convergence of the solutions to yield better approximation of compressible fluid by incompressible one. Therefore it is interesting to study sufficient conditions for the convergence to be strong.

Strong convergence of the velocity was established in [5, III, §8] for the solutions of "compressible" system arising in the method of artificial compressibility. It was also proved that the gradient of the pressure converges weakly, but strong convergence of the pressure was not examined.

In some situations the convergence to the incompressible limit cannot be strong. In [6] there was derived a *necessary* condition of strong convergence

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of *classical* solutions to equations of compressible fluid when compressibility tends to zero. The condition represents a restriction on the initial condition for the equations of incompressible fluid and this restriction cannot be satisfied in general case.

In this paper we study *weak* solutions to *linearized* equations of compressible fluid. The reason for this is twofold. First, by the present moment only existence of weak solutions (on arbitrary time interval) to the full equations of compressible fluid is established [2], and there is well-known regularity problem for these solutions. Second, the proper treatment of nonlinear terms not only would be hard technically but might perhaps obscure the effect of low compressibility on the solutions, which is the main subject of this paper. (Also note that the linearized equations of compressible fluid are of interest on they own. For instance, spectral properties of the operator corresponding to linearized steady equations of compressible fluid were examined in [7].)

Linearized equations of compressible fluid motion were studied in [8, 9]. Estimate of strong generalized solution to initial–boundary value problem for these equations in a bounded three-dimensional domain was derived in [9], and existence of strong generalized solution to Cauchy problem in the whole space for them was established in [8]. It appears that existence of weak solutions to these equations has not been addressed.

In this paper we derive existence, uniqueness and estimates of weak solutions to the initial–boundary value problem for linearized equations of compressible fluid. We examine convergence of the solutions to the incompressible limit when compressibility tends to zero. Briefly, we prove that

- in general case the velocity field converges *weakly*;
- if initial condition for the velocity is solenoidal then the velocity converges *strongly* and the pressure converges $*$-*weakly*;
- if, in addition, the initial condition for the pressure is compatible with the initial value of the pressure in the incompressible fluid then the convergence of the pressure is *strong*.

2 Notation and Preliminaries

2.1 Common Functional Spaces

Let $E \subset \mathbb{R}^n$ be a bounded domain $n \in \mathbb{N}$. We will use the following standard spaces:

- $L^p(E)$ is the Lebesgue space of real-valued functions on E, $1 \le p \le \infty$;
- $\widehat{L}^p(E) = \big\{ u \big|\ u \in L^p(E),\ \int_E u\,dx = 0 \big\}$;
- $H^s(E) = W^{s,p}(E)$ is the Sobolev space of real-valued functions whose weak derivatives up to order $s \in \mathbb{N}$ belong to $L^p(E)$;

- $\mathcal{D}'(E)$ is the space of distributions on E; $\mathcal{D}(E)$ is the space of test functions on E;
- $C_0^\infty(E)$ is the space of smooth real-valued functions on E with compact support;
- $H_0^1(E)$ denotes the closure of $C_0^\infty(E)$ in $H^1(E)$-norm;
- $H^{-1}(E)$ denotes the dual space of $H_0^1(E)$;

(For $E = (a, b) \subset \mathbb{R}$ we will omit undue brackets, i.e. $L^2(a, b) = L^2((a, b))$.)

For $\mathbb{R}^k$-valued functions ($k \in \mathbb{N}$) we will use Cartesian products of these spaces, e.g. $L^2(E)^k$. Since $H^{-1}(D)^k$ is linearly and continuously isomorphic to the dual space of $H_0^1(D)^k$, let $H^{-1}(D)^k$ *denote* the latter.

Let $D \subset \mathbb{R}^d$ be a bounded domain with a partially-smooth boundary ∂D, $d \in \mathbb{N}$, $d \geq 2$. Let $Q_T = D \times (0, T)$, where $T > 0$.

Vector-valued functions will be denoted by bold letters. For such functions we will use the following spaces:

- $\dot{J}(D) = \{\mathbf{u} \big| \ \mathbf{u} \in C_0^\infty(D)^d, \ \operatorname{div} \mathbf{u} = 0\}$;
- $J(D)$ is the closure in $L^2(D)$-norm of $\dot{J}(D)$ (cf. [10]);
  (This space is often also denoted as $H(D)$.)
- $V(D)$ is the closure in $H_0^1(D)^d$-norm of $\dot{J}(D)$.

2.2 Scalar Products and Duality

It is well-known that the space $H_0^1(D)^k$ ($k \in \mathbb{N}$) is a Hilbert space with respect to the dot product

$$((\mathbf{u}, \mathbf{v})) = \int_D (\nabla \otimes \mathbf{u}) : (\nabla \otimes \mathbf{v}) \, dx \equiv \int_D (\partial_i u_k)(\partial_i v_k) \, dx$$

where $\{\mathbf{u}, \mathbf{v}\} \subset H_0^1(D)^k$ are vector-valued functions (here and further Einstein summation convention is used).

Let $(\cdot, \cdot)$, depending on the context, denote the standard dot product in $\mathbb{R}^k$ or the dot product in $L^2(D)^k$, i.e. $(\mathbf{u}, \mathbf{v}) = \int_D u_i v_i \, dx$.

Let $\|\cdot\|_X$ denote the norm of a Banach space X (with dual space X^*) and let $\langle \cdot, \cdot \rangle_X$ denote the duality brackets for the pair (X^*, X). Let $(x_n) \subset X$, $x \in X$. It is convenient to use the following notation:

- $x_n \rightharpoonup x$ means that the sequence (x_n) converges to x weakly;
- $x_n \rightharpoondown x$ means that the sequence (x_n) converges to x $*$-weakly.

We will also use the following short-hand notation for norms (see [5]):

$$|\cdot| \equiv \|\cdot\|_{L^2(D)^k}$$
$$\|\cdot\| \equiv \|\cdot\|_{H_0^1(D)^k}$$

where the value of $k \in \mathbb{N}$ is defined by the context.

2.3 Differential Operators

It is known that the operators ∇, Δ and div, defined on smooth functions with compact support (i.e. functions from $C_0^\infty(D)^k$, $k \in \mathbb{N}$), can be extended by continuity to bounded linear operators $\nabla\colon L^2(D) \to H^{-1}(D)^d$, $\Delta\colon H_0^1(D)^d \to H^{-1}(D)^d$ and $\operatorname{div}: L^2(D)^d \to H^{-1}(D)$, defined by

$$\begin{aligned} \nabla\colon &\quad p \mapsto -(p, \operatorname{div}\cdot), \\ \Delta\colon &\quad \mathbf{u} \mapsto -((\mathbf{u}, \cdot)), \\ \operatorname{div}: &\quad \mathbf{u} \mapsto -(\mathbf{u}, \nabla\cdot). \end{aligned}$$

2.4 Spaces of Banach Space Valued Functions

Consider an arbitrary closed interval $[0, T]$ where $T > 0$. Let X be a Banach space and $s \in \mathbb{N}$. Let $L^p(0, T; X)$ and $W^{s,p}(0, T; X)$ denote accordingly Lebesgue–Bochner and Sobolev–Bochner spaces of X-valued functions of real variable $t \in [0, T]$ (see, e.g., [11]), $1 \le p \le \infty$. Let f_t denote the weak derivative of a function $f \in W^{1,p}(0, T; X)$ with respect to t.

Now we recall the following well-known property of Sobolev space of Banach space valued functions (see, e.g., [11]):

Proposition 2.1. *Every $u \in W^{1,2}(0, T; X)$ can be redefined on a subset $M \subset [0, T]$ of zero measure so that $u \in C(0, T; X)$.* (Here and further $C(0, T; X)$ denotes the space of continuous functions $f\colon [0, T] \to X$.)

Remark 2.2. Let $\overline{u}$ denote the redefined version of u. Then $\forall \varphi \in C^\infty([0, T])$, $\forall a, b \in [0, T]$

$$\int_a^b u_t \varphi \, dt = \overline{u}\varphi|_a^b - \int_a^b u \varphi_t \, dt$$

Moreover, the mapping $t \mapsto \|\overline{u}(t)\|_H^2$ is absolutely continuous with

$$\partial_t \|\overline{u}(t)\|_H^2 = 2(u_t(t), u(t))_H.$$

for a.e. $t \in [0, T]$.

Let X be a reflexive Banach space (with dual space X^*) and let H be a Hilbert space for which there exists a linear bounded dense embedding $\kappa\colon X \to H$. Let $\pi\colon H \to X^*$ be the embedding given by $\pi\colon h \mapsto (h, \kappa(\cdot))_H$, where $(\cdot, \cdot)_H$ is the dot product in H. Then embeddings π and $\imath = \pi \circ \kappa$ are linear, bounded and dense. Triple (X, H, X^*) (with embeddings $\kappa, \pi, \imath$) is said to be an *evolution triple* [11]. For given evolution triple let

$$\widetilde{W}^{1,2}(0, T; X) = \left\{ f \middle| \; f \in L^2(0, T; X), \; \imath(f) \in W^{1,2}(0, T; X^*) \right\}.$$

This space is referred to as *Sobolev–Lions* space [12]. It is a reflexive Banach space with norm given by

$$\|f\|_{\widetilde{W}^{1,2}(0,T;X)} = \|f\|_{L^2(0,T;X)} + \|\imath(f)_t\|_{L^2(0,T;X^*)}.$$

Embedding $\imath$ is often omitted and the space $\widetilde{W}^{1,2}(0,T;X)$ is then introduced as the space of functions belonging to $L^2(0,T;X)$ whose weak derivative belongs to $L^2(0,T;X^*)$. In this paper $\imath$, κ and π will be omitted when they are not the subject matter.

The introduced space has the following property (see, e.g., [11]):

Proposition 2.3. *Every $u \in \widetilde{W}^{1,2}(0,T;X)$ can be redefined on a subset $M \subset [0,T]$ of zero measure so that $u \in C(0,T;H)$.*

Remark 2.4. Let $\overline{u}$ denote the redefined version of u. Then $\forall a, b \in [0,T]$, $\forall v \in X$ and $\forall \varphi \in C^\infty([0,T])$

$$\int_a^b \langle u_t, v\rangle \,\varphi \,dt = (\overline{u}(t)\varphi(t), v)_H\Big|_a^b - \int_a^b (u,v)_H \varphi_t \,dt.$$

Consequently, *if $\forall v \in X$ and $\forall \varphi \in C_0^\infty([0,T))$*

$$\int_0^T \langle u_t, v\rangle \,\varphi \,dt = -(u_0\varphi(0), v)_H - \int_0^T (u,v)_H \varphi_t \,dt,$$

where $u_0 \in H$, then $\overline{u}(0) = u_0$. (Similar procedure can be used to identify the initial value of a function from $W^{1,2}(0,T;H)$.)

Remark 2.5. The mapping $t \mapsto \|\overline{u}(t)\|_H^2$ is absolutely continuous with

$$\partial_t \|\overline{u}(t)\|_H^2 = 2\,\langle u_t(t), u(t)\rangle$$

for a.e. $t \in [0,T]$.

In this paper we consider evolution triples $(H_0^1(D)^k, L^2(D)^k, H^{-1}(D)^k)$ ($k \in \mathbb{N}$) and $(V(D), J(D), V(D)^*)$. For both of them the embeddings κ, π and $\imath$ are given by $\kappa\colon \mathbf{u} \mapsto \mathbf{u}$ (natural embedding), $\pi\colon \mathbf{u} \mapsto \int_D (\mathbf{u}, \cdot)\,dx$ and $\imath = \pi \circ \kappa$.

The following theorem describes the space which is dual to a Lebesgue–Bochner space [11]:

Proposition 2.6. *If X is a reflexive Banach space, then*

$$L^p(0,T;X)^* = L^{p'}(0,T;X^*),$$

where $1 \le p < \infty$, $1/p + 1/p' = 1$ and duality is given by $\langle f, g\rangle = \int_0^T fg\,dt$, $f \in L^{p'}(0,T;X^*)$, $g \in L^p(0,T;X)$.

Different constants which are not dependent on the principal parameters (such as initial conditions) will be denoted by the same letter C. The dependence of such constant on some parameter will be indicated in the subscript.

2.5 Auxiliary Inequalities

Let us recall two well-known statements:

Proposition 2.7. *Let $a \geq 0$, $b \geq 0$ and J be real numbers. If $J^2 \leq a + bJ$ then $J \leq b + \sqrt{a}$.*

Proposition 2.8 (Gronwall's inequality)**.** *Let I be an absolutely continuous nonnegative function of a variable $t \in [0, T]$ and let $\varphi, \psi \in L^1(0, T)$ be nonnegative functions. If the derivative $I'(t)$ satisfies*

$$I'(t) \leq \varphi(t)I(t) + \psi(t) \quad \text{a.e. on } [0, T],$$

then for a.e. $t \in [0, T]$

$$I(t) \leq e^{\int_0^t \varphi(\tau)\, d\tau} \left(I(0) + \int_0^t \psi(\tau)\, d\tau \right).$$

The proof of Proposition 2.7 is elementary and the proof of Proposition 2.8 can be found e.g. in [13].

We will use the following mix of Lemmas 2.7 and 2.8:

Lemma 2.9. *Let I and J be absolutely continuous nonnegative functions of a variable $t \in [0, T]$, $J \in L^2(0, T)$. Let $a, c \in L^1(0, T)$ and $b \in L^2(0, T)$ be nonnegative functions. If for a.e. $t \in [0, T]$*

$$I'(t) + J^2(t) \leq a(t)I(t) + b(t)J(t) + c(t) \tag{2.1}$$

then

$$\|J\|_{L^2(0,T)} \leq C_a \left(\sqrt{I(0)} + \sqrt{\|c\|_{L^1(0,T)}} + \|b\|_{L^2(0,T)} \right),$$

$$\|I\|_{L^\infty(0,T)} \leq C_a \left(I(0) + \|c\|_{L^1(0,T)} + \|b\|^2_{L^2(0,T)} \right),$$

where constant C_a depends only on $A = \|a\|_{L^1(0,T)}$

Proof. From (2.1) for a.e. $t \in [0, T]$

$$I'(t) \leq a(t)I(t) + b(t)J(t) + c(t)$$

and then, by Proposition 2.8 and Cauchy–Bunyakovsky inequality,

$$I(t) \leq \exp\left(\int_0^t a\, d\tau \right) \left(I(0) + \int_0^t (bJ + c) d\tau \right) \leq$$
$$\leq e^{\|a\|_{L^1(0,t)}} \left(I(0) + \|b\|_{L^2(0,t)} \|J\|_{L^2(0,t)} + \|c\|_{L^1(0,t)} \right)$$

and hence

$$\|I\|_{L^\infty(0,T)} \leq e^{\|a\|_{L^1(0,T)}} \left(\|b\|_{L^2(0,T)} \|J\|_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) \right).$$

Integrating the inequality $J^2 \leq aI + bJ + c - I'$ and noting that $I(T) \geq 0$ we obtain

$$\int_0^T J^2 dt \leq \int_0^T aI\, d\tau + \int_0^T bJ\, d\tau + \int_0^T c\, d\tau + I(0) \leq$$
$$\leq \|I\|_{L^\infty(0,T)} \|a\|_{L^1(0,T)} + \|b\|_{L^2(0,T)} \|J\|_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) \leq$$
$$\leq C_a \left(\|b\|_{L^2(0,T)} \|J\|_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) \right) \tag{2.2}$$

where
$$C_a = 1 + \|a\|_{L^1(0,T)} e^{\|a\|_{L^1(0,T)}}.$$
Applying Proposition 2.7 to (2.2) we get

$$\|J\|_{L^2(0,T)} \le C_a \left(\sqrt{\|c\|_{L^1(0,T)}} + \sqrt{I(0)} + \|b\|_{L^2(0,T)} \right)$$

Finally, by Young's inequality

$$\begin{aligned} \|I\|_{L^\infty(0,T)} \le e^{\|a\|_{L^1(0,T)}} \left(\frac{\|b\|^2_{L^2(0,T)} + \|J\|^2_{L^2(0,T)}}{2} + \|c\|_{L^1(0,T)} + I(0) \right) \le \\ \le e^{\|a\|_{L^1(0,T)}} \Big(\|b\|^2_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) + \\ + \frac{3}{2} C_a^2 \left(\|b\|^2_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) \right) \Big) = \\ = \tilde{C}_a \left(\|b\|^2_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) \right), \end{aligned}$$

where
$$\tilde{C}_a = e^{\|a\|_{L^1(0,T)}} \left(1 + \frac{3}{2} C_a^2 \right). \qquad \square$$

2.6 Properties of Special Subspaces

Let $G(D)$ denote the orthogonal complement of $J(D)$ in $L^2(D)^d$. Let P_J and P_G denote the orthogonal projectors of $L^2(D)^d$ onto $J(D)$ and $G(D)$ respectively. Projector P_J is referred to as *Leray projector*.

Let
$$V(D)^\perp = \left\{ f \middle| \; f \in H^{-1}(D)^d, \; f(\mathbf{u}) = 0, \; \forall \mathbf{u} \in V(D) \right\}.$$
It is clear that functionals represented by ∇p, $p \in L^2(D)$, belong to $V(D)^\perp$. To show that any functional from the latter space can be represented is such form, the following statements are needed (see [14] and [5, I, §1]):

Proposition 2.10. *A functional $f \in H^{-1}(D)^d$ is representable in the form*

$$f = \nabla p$$

for some $p \in L^2(D)$ if and only if

$$\langle f, \mathbf{u} \rangle = 0 \quad \forall \mathbf{u} \in \dot{J}(D).$$

Proposition 2.11. *If $p \in \widehat{L}^2(D)$ then*

$$\|p\|_{\widehat{L}^2(D)} \le C \|\nabla p\|_{H^{-1}(D)^d},$$

where the constant C depends only on the domain D.

Propositions 2.10 and 2.11 allow us to introduce an operator, which is inverse to ∇:

Lemma 2.12. *The operator* $\nabla\colon \widehat{L}^2(D) \to V(D)^\perp$ *has a bounded inverse* ∇^{-1}.

Proof. For $f \in V(D)^\perp$ let $p_1 = p - \int_D p\,dx$, where p is given by Proposition 2.10. If there exists $p_2 \in \widehat{L}^2(D)$ such that $f = \nabla p_2$ then $\nabla(p_1 - p_2) = 0$ and from Proposition 2.11 $p_1 - p_2 = 0$, therefore we can introduce the operator $\nabla^{-1}\colon f \mapsto p_1$ (which is clearly linear).

It follows from Proposition 2.11 that $\|p_1\|_{\widehat{L}^2(D)} \le C\|f\|_{H^{-1}(D)}$ and therefore ∇^{-1} is bounded. ◻

Later we will use the operator which was first constructed by M.E. Bogovskii [15]:

Proposition 2.13. *Suppose that the domain D is star-shaped with respect to some ball. Then there exists a bounded linear operator $\mathcal{B}\colon \widehat{L}^2(D) \to H_0^1(D)^d$ such that $\forall f \in \widehat{L}^2(D)$ the field $\mathbf{u} = \mathcal{B}(f)$ satisfies*

$$\operatorname{div}\mathbf{u} = f \quad \textit{(a.e. in } D\textit{)}.$$

3 Linearized Equations of Low Compressible Fluid Motion

Consider viscous barotropic fluid with state equation $\rho = F(p)$, where ρ and p are density and pressure respectively. Let us linearize this equation near some reference pressure p_{ref}:

$$\rho = F(p) \approx F(p_{ref}) + F'(p_{ref})(p - p_{ref}).$$

For brevity assume that $p_{ref} = 0$ and denote $F(0) = \rho_0 > 0$, $F'(0) = \alpha$. From the physical point of view $1/\alpha = \frac{\partial p}{\partial \rho} = c^2 > 0$, where c is the speed of sound. Ultimately we have $\rho = \rho_0 + \alpha p$.

We are going to study the following equations:

$$\begin{gathered}
\rho_t + \rho_0 \operatorname{div}\mathbf{u} = 0, \\
\rho_0 \mathbf{u}_t + \nabla p = \mu\Delta\mathbf{u} + \eta\nabla\operatorname{div}\mathbf{u} + \rho\mathbf{f}, \\
\rho = \rho_0 + \alpha p.
\end{gathered} \tag{3.1}$$

Here

$\rho = \rho(x,t) \quad (\rho(x,t) \in \mathbb{R}), \qquad p = p(x,t) \quad (p(x,t) \in \mathbb{R}),$

$x \in D \subset (0,T)$, $t \in (0,T) \subset \mathbb{R}$,

$\mathbf{u} = \mathbf{u}(x,t)$ is the velocity ($\mathbf{u}(x,t) \in \mathbb{R}^d$),

$\mathbf{f} = \mathbf{f}(x,t)$ is the external force (per unit volume).

The coefficients of viscosity $\mu > 0$ and $\eta \geq 0$ are constant throughout the paper. The constant $\alpha > 0$ is referred to as the *compressibility factor* [6].

Equations (3.1) arise as the linearisation of the equations of compressible fluid motion near some solution $\{\boldsymbol{v}, q\}$ to the corresponding equations of incompressible fluid. The convective terms $(\boldsymbol{v}, \nabla)\alpha p$ and $(\boldsymbol{v}, \nabla)\mathbf{u}$ are omitted for the sake of technical simplicity.

Let us consider the following initial–boundary conditions for the equations (3.1):

$$\mathbf{u}|_{\partial D} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad p|_{t=0} = p_0. \tag{3.2}$$

It is clear that the solution to the problem (3.1), (3.2) depends on the compressibility factor α:

$$\{\mathbf{u}, p\} = \{\mathbf{u}_\alpha, p_\alpha\}.$$

Our goal is to study the convergence of these solutions when compressibility tends to zero, i.e. $\alpha \to 0$. But before doing this, we need to state (and prove) the existence and uniqueness theorems for the problem (3.1), (3.2) and for the corresponding incompressible system.

3.1 Existence and Uniqueness of Weak Solutions

Consider a non-homogeneous form of the problem (3.1), (3.2):

$$\begin{gathered} \rho_t + \rho_0 \mathrm{div}\, \mathbf{u} = \sigma, \quad \rho = \alpha p, \\ \rho_0 \mathbf{u}_t + \nabla p = \mu \Delta \mathbf{u} + \eta \nabla \mathrm{div}\, \mathbf{u} + \rho \mathbf{f} + \mathbf{s}, \end{gathered} \tag{3.3}$$

$$\mathbf{u}|_{\partial D} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad p|_{t=0} = p_0. \tag{3.4}$$

Setting $\sigma = 0$ and $\mathbf{s} = \rho_0 \mathbf{f}$ we clearly obtain the problem (3.1), (3.2).

Assume that

$$\begin{gathered} \sigma = L^2(0, T; L^2(D)), \quad \mathbf{s} \in L^2(0, T; H^{-1}(D)^d), \\ \mathbf{f} \in L^\infty(Q_T)^d, \quad \mathbf{u}_0 \in L^2(D)^d, \quad p_0 \in L^2(D). \end{gathered}$$

Definition 3.1. A pair

$$\{\mathbf{u}, p\} \in L^2(0, T; H_0^1(D)^d) \times L^2(0, T; L^2(D))$$

is called a *weak solution* to the problem (3.3), (3.4), if for all $\varphi \in C_0^\infty([0, T))$, $\mathbf{v} \in H_0^1(D)^d$ and $r \in L^2(D)$

$$-\int_0^T (\rho, r)\, \varphi_t \, dt - (\alpha p_0, r)\varphi(0) + \int_0^T (\rho_0 \mathrm{div}\, \mathbf{u}, r)\, \varphi \, dt = \int_0^T (\sigma, r)\, \varphi \, dt, \tag{3.5}$$

$$\begin{aligned} -\int_0^T \rho_0(\mathbf{u}, \mathbf{v})\, \varphi_t \, dt - \rho_0(\mathbf{u}_0, \mathbf{v})\varphi(0) - \int_0^T (p, \mathrm{div}\, \mathbf{v})\, \varphi \, dt = \\ = -\mu \int_0^T ((\mathbf{u}, \mathbf{v}))\, \varphi \, dt - \eta \int_0^T (\mathrm{div}\, \mathbf{u}, \mathrm{div}\, \mathbf{v})\, \varphi \, dt + \\ + \int_0^T (\rho \mathbf{f}, \mathbf{v})\, \varphi \, dt + \int_0^T \langle \mathbf{s}, \mathbf{v} \rangle\, \varphi \, dt, \end{aligned} \tag{3.6}$$

where $\rho = \alpha p$. (See subsection 2.2 for the notation.)

Remark 3.2. From (3.5) and (3.6) it follows that

$$\mathbf{u} \in \widetilde{W}^{1,2}(0,T; H_0^1(D)^d),$$
$$p \in W^{1,2}(0,T; L^2(D))$$

and the equations (3.3) hold for a.e. $t \in [0,T]$ in sense of notation, introduced in subsections 2.3 and 2.4. Consequently, for a.e. $t \in [0,T]$

$$(\rho_t + \rho_0 \operatorname{div} \mathbf{u}, p) = (\sigma, p),$$
$$\rho_0 \langle \mathbf{u}_t, \mathbf{u} \rangle - (p, \operatorname{div} \mathbf{u}) = \langle \mu \Delta \mathbf{u} + \eta \nabla \operatorname{div} \mathbf{u} + \rho \mathbf{f} + \mathbf{s}, \mathbf{u} \rangle$$

Due to Propositions 2.1 and 2.3 the functions $\mathbf{u}$ and p can be redefined so that they have well-defined values $\overline{\mathbf{u}}(t)$ and $\overline{p}(t)$ at each $t \in [0,T]$. Integrating the equations above with respect to t using Remarks 2.2 and 2.5 we obtain the *energy equality*:

$$\frac{1}{2} \left(\rho_0 |\overline{\mathbf{u}}|^2 + \frac{\alpha}{\rho_0} |\overline{p}|^2 \right) \Big|_\xi^\tau + \int_\xi^\tau (\mu \|\mathbf{u}\|^2 + \eta |\operatorname{div} \mathbf{u}|^2)\, dt =$$
$$= \int_\xi^\tau (\alpha p \mathbf{f}, \mathbf{u})\, dt + \int_\xi^\tau \langle \mathbf{s}, \mathbf{u} \rangle\, dt + \int_\xi^\tau \frac{1}{\rho_0} (\sigma, p)\, dt, \quad (3.7)$$

where $\xi, \tau \in [0,T]$ are arbitrary.

By Remark 2.4, from equations (3.5), (3.6) we conclude that $\overline{\mathbf{u}}(0) = \mathbf{u}_0$ and $\overline{p}(0) = p_0$.

Definition 3.1 is equivalent to the following one:

Definition 3.3. A pair

$$\{\mathbf{u}, p\} \in L^2(0,T; H_0^1(D)^d) \times L^2(0,T; L^2(D))$$

is called a *weak solution* to the problem (3.3), (3.4), if the equations (3.3) hold in sense of distributions on Q_T, and the initial values of $\mathbf{u}$ and p are $\mathbf{u}_0$ and p_0 respectively.

Proof of equivalence of 3.1 and 3.3. To prove the implication 3.3 $\Rightarrow$ 3.1 consider $\mathbf{v} \in H_0^1(D)^d$ and $r \in L^2(D)$. There exist $(\mathbf{v}_n) \subset C_0^\infty(D)^d$ and $(r_n) \subset C_0^\infty(D)$ such that $\mathbf{v}_n \to \mathbf{v}$ and $r_n \to r$ in $H_0^1(D)^d$ and $L^2(D)$ respectively. Then, for each $\varphi \in C_0^\infty(0,T)$

$$\mathbf{\Phi}_n(x,t) = \mathbf{v}_n(x)\varphi(t) \in \mathcal{D}(Q_T)^d,$$
$$\Phi_n(x,t) = r_n(x)\varphi(t) \in \mathcal{D}(Q_T).$$

Integrating by parts the distributional formulation of the equations (3.3) and passing to the limit when $n \to \infty$ we obtain (3.5) and (3.6) with $\varphi \in C_0^\infty(0,T)$.

Then, from Propositions 2.1 and 2.3 it follows that (3.5) and (3.6) hold with arbitrary $\varphi \in C_0^\infty([0,T))$.

The implication 3.1 ⇒ 3.3 follows from the fact that the set

$$\left\{ \sum_{k=1}^{N} \varphi_k(t) r_k(x) \,\middle|\, N \in \mathbb{N},\ \varphi_k \in \mathcal{D}(0,T),\ r_k \in \mathcal{D}(D),\ k = 1..N \right\}$$

is dense in $\mathcal{D}(Q_T)$ (see e.g. [16]). □

Now we are ready to state the main result of this section:

Theorem 3.4. *The problem* (3.3), (3.4) *has a unique weak solution* $\{\mathbf{u}, p\}$. *For this solution the following estimates are valid:*

$$\begin{aligned} &\|\mathbf{u}\|_{L^2(0,T;H_0^1(D)^d)} + \|\mathbf{u}\|_{L^\infty(0,T;L^2(D)^d)} + \\ &\qquad + \sqrt{\alpha}\|p\|_{L^\infty(0,T;L^2(D))} \le C_{\mathbf{f},\alpha} E, && (3.8) \\ &\|\mathbf{u}\|_{\widetilde{W}^{1,2}(0,T;H_0^1(D)^d)} \le \frac{1}{\sqrt{\alpha}} \tilde{C}_{\mathbf{f},\alpha} E, && (3.9) \end{aligned}$$

where

$$E \equiv \|\mathbf{u}_0\|_{L^2(D)^d} + \sqrt{\alpha}\|p_0\|_{L^2(D)} + \\ + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)}$$

and the constants $C_{\mathbf{f},\alpha}, \tilde{C}_{\mathbf{f},\alpha}$ *depend only on* $\|\mathbf{f}\|_{L^\infty(Q_T)^d}$ *and* α. *For fixed* $\mathbf{f}$ *these constants are bounded when* $\alpha \to 0$. *Moreover, if* $p_0 \in \widehat{L}^2(D)$ *and* $\sigma \in L^2(0,T;\widehat{L}^2(D))$, *then*

$$p \in W^{1,2}(0,T;\widehat{L}^2(D)).$$

Proof. We will use Faedo–Galerkin method. Let $\{\mathbf{e}_j\}_{j=1}^\infty$ be an orthonormal with respect to $((\cdot,\cdot))$ basis in $H_0^1(D)^d$ and let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis in $L^2(D)$. Then, since $H_0^1(D)^d$ is dense in $L^2(D)$, for $m \in \mathbb{N}$ there exist

$$p_m^0 = \sum_{j=1}^{m} \tilde{c}_{0,j} e_j, \quad \tilde{c}_{0,j} = (p_0, e_j),$$

$$\mathbf{u}_m^0 = \sum_{j=1}^{m} c_{0,j}^{(m)} \mathbf{e}_j, \quad c_{0,j}^{(m)} \in \mathbb{R},$$

such that $p_m^0 \to p_0$ in $L^2(D)$ and $\mathbf{u}_m^0 \to \mathbf{u}_0$ in $L^2(D)^d$ when $m \to \infty$.

Let us look for the solution in the form

$$\mathbf{u}_m = \sum_{j=1}^{m} c_j^{(m)}(t) \mathbf{e}_j, \quad p_m = \sum_{j=1}^{m} \tilde{c}_j^{(m)}(t) e_j,$$

where the functions $c_j^{(m)}(t)$ and $\tilde{c}_j^{(m)}(t)$ are absolutely continuous. Consider the following "approximate" system:

$$(\rho_{m,t} + \rho_0 \mathrm{div}\, \mathbf{u}_m, e_i) = (\sigma, e_i) \quad \text{a.e. on } [0,T],$$
$$\langle \rho_0 \mathbf{u}_{m,t} + \nabla p_m, \mathbf{e}_i \rangle = \langle \mu \Delta \mathbf{u}_m + \eta \nabla \mathrm{div}\, \mathbf{u}_m + \rho_m \mathbf{f} + \mathbf{s}, \mathbf{e}_i \rangle \quad \text{a.e. on } [0,T],$$
$$i = 1..m, \quad \rho_m = \alpha p_m,$$
$$\mathbf{u}_m|_{t=0} = \mathbf{u}_m^0, \quad p_m|_{t=0} = p_m^0.$$

This system represents a Cauchy problem for a linear system of first order differential equations. Under our assumptions for each m it has a unique solution $\{c_j^{(m)}(t), \tilde{c}_j^{(m)}(t)\}_{j=1}^m$ such that

$$\{u_m, p_m\} \in W^{1,2}(0,T; H_0^1(D)^d) \times W^{1,2}(0,T; L^2(D)).$$

Since $\mathbf{u}_m(t) \in \mathrm{span}\{\mathbf{e}_j\}_{j=1}^m$ and $p_m(t) \in \mathrm{span}\{e_j\}_{j=1}^m$, the following equations hold a.e. on $[0,T]$:

$$(\rho_{m,t} + \rho_0 \mathrm{div}\, \mathbf{u}_m, p_m) = (\sigma, p_m),$$
$$(\rho_0 \mathbf{u}_{m,t}, \mathbf{u}_m) - (p_m, \mathrm{div}\, \mathbf{u}_m) = \langle \mu \Delta \mathbf{u}_m + \eta \nabla \mathrm{div}\, \mathbf{u}_m + \rho_m \mathbf{f} + \mathbf{s}, \mathbf{u}_m \rangle$$

Hence, recalling the notation from subsection 2.2, by Remarks 2.2 and 2.5

$$\frac{1}{2}\left(\rho_0 |\mathbf{u}_m|^2 + \frac{\alpha}{\rho_0}|p_m|^2\right)_t + \mu \|\mathbf{u}_m\|^2 + \eta |\mathrm{div}\, \mathbf{u}_m|^2 =$$
$$= (\alpha p_m \mathbf{f}, \mathbf{u}_m) + \langle \mathbf{s}, \mathbf{u}_m \rangle + \frac{1}{\rho_0}(\sigma, p_m). \quad (3.10)$$

Let us make some auxiliary estimates. First, from Young's inequality,

$$\frac{\sqrt{\alpha}}{\rho_0}(\sqrt{\alpha} p_m \mathbf{f}, \rho_0 \mathbf{u}_m) \leq \frac{1}{2}\frac{\sqrt{\alpha}}{\rho_0}\|\mathbf{f}\|_{L^\infty(Q_T)^d}\left(\alpha |p_m|^2 + \rho_0^2 |\mathbf{u}_m|^2\right)$$

and $(\sigma, p_m) \leq \frac{1}{2}\left(\alpha |p_m|^2 + \frac{1}{\alpha}|\sigma|^2\right)$. Second, $\langle \mathbf{s}, \mathbf{u}_m \rangle \leq \|\mathbf{s}\|_{H^{-1}(D)^d}\|\mathbf{u}_m\|$. Then, denoting

$$I(t) = \frac{1}{2}\left(\rho_0 |\mathbf{u}_m(t)|^2 + \frac{\alpha}{\rho_0}|p_m(t)|^2\right), \quad J(t) = \sqrt{\mu}\|\mathbf{u}_m(t)\|,$$
$$a = 1 + \sqrt{\alpha}\|\mathbf{f}\|_{L^\infty(Q_T)^d}, \quad b(t) = \frac{1}{\sqrt{\mu}}\|\mathbf{s}(t)\|_{H^{-1}(D)^d}, \quad c(t) = \frac{1}{2\rho_0 \alpha}|\sigma(t)|^2$$

from (3.10) we obtain

$$I' + J^2 \leq aI + bJ + c$$

From this inequality, by Lemma 2.9, we have

$$\|J\|_{L^2(0,T)} \leq C_{\mathbf{f},\alpha}\left(|\mathbf{u}_m^0| + \sqrt{\alpha}|p_m^0| + \right.$$
$$\left. + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)}\right),$$

$$\|I\|_{L^\infty(0,T)} \le C_{\mathbf{f},\alpha} \left(|\mathbf{u}_m^0|^2 + \alpha |p_m^0|^2 + \right. \\ \left. + \frac{1}{\alpha} \|\sigma\|^2_{L^2(0,T;L^2(D))} + \|\mathbf{s}\|^2_{L^2(0,T;H^{-1}(D)^d)} \right).$$

From the proof of Lemma 2.9 it follows that the constant $C_{\mathbf{f},\alpha}$ depends only on $\|\mathbf{f}\|_{L^\infty(Q_T)^d}$ and α, and for each $\mathbf{f}$ this constant is bounded when $\alpha \to 0$.

Thus we have proved that estimate (3.8) holds for the "approximate solutions" $\mathbf{u}_m$, p_m. Since any weak solution to the problem (3.3), (3.4) satisfies the energy equality (3.7), the arguments above imply that (3.8) holds for each weak solution to the problem (3.3), (3.4) as well.

By Alaoglu–Bourbaki theorem (and Proposition 2.6), there exists a subsequence (for brevity not renumbered) such that

$$\begin{aligned} \mathbf{u}_m &\rightharpoonup \mathbf{u} \ \text{ in } L^2(0,T;H_0^1(D)^d), \\ \mathbf{u}_m &\rightharpoondown \mathbf{u} \ \text{ in } L^\infty(0,T;L^2(D)^d), \\ p_m &\rightharpoondown p \ \text{ in } L^\infty(0,T;L^2(D)) \end{aligned}$$

and the limits $\mathbf{u}$ and p satisfy (3.8).

Now let $\varphi \in C_0^\infty([0,T))$ and for $r \in L^2(D)$ denote

$$F_m(r) \equiv \int_0^T (-\rho_m \varphi_t + (\rho_0 \operatorname{div} \mathbf{u}_m - \sigma)\varphi, r)\, dt - (\alpha p_m(0), r)\varphi(0),$$

It is clear that (F_m) is bounded in $L^2(D)^*$ and for each $r \in \operatorname{span}\{e_j\}_{j=1}^m$ $F_m(r) = 0$. Then, since $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(D)$,

$$\lim_{m\to\infty} F_m(r) = 0.$$

On the other hand,

$$\lim_{m\to\infty} F_m(r) = F(r) \equiv \int_0^T (-\rho\varphi_t + (\rho_0 \operatorname{div} \mathbf{u} - \sigma)\varphi, r)\ dt - (\alpha p_0, r)\varphi(0),$$

where $\rho = \alpha p$. Thus we have shown that (3.5) holds. Similar arguments for the sequence of the functionals

$$G_m(\mathbf{v}) \equiv \int_0^T (-\rho_0 \mathbf{u}_m \varphi_t, \mathbf{v})\, dt - \int_0^T (p_m, \operatorname{div} \mathbf{v})\varphi\, dt - \rho_0(\mathbf{u}_m(0), \mathbf{v})\varphi(0) - \\ - \int_0^T \langle \mu\Delta\mathbf{u}_m + \eta\nabla \operatorname{div} \mathbf{u}_m + \rho_m \mathbf{f} + \mathbf{s}, \mathbf{v}\rangle\, \varphi\, dt, \qquad \mathbf{v} \in H_0^1(D)^d$$

imply that (3.6) holds.

From (3.5) and (3.6) we conclude that a.e. on $[0,T]$

$$\begin{aligned} |p_t| &\le \frac{1}{\alpha}\left(|\rho_0 \operatorname{div} \mathbf{u}| + |\sigma|\right), \\ \|\mathbf{u}_t\|_{H^{-1}(D)^d} &\le \|-\nabla p + \mu\Delta\mathbf{u} + \eta\nabla\mathbf{u} + \alpha p\mathbf{f} + \mathbf{s}\|_{H^{-1}(D)^d}/\rho_0, \end{aligned}$$

hence $\mathbf{u}$ satisfies (3.9).

If the problem (3.3), (3.4) had two weak solutions $\{\mathbf{u}^1, p^1\}$ and $\{\mathbf{u}^2, p^2\}$, then the difference $\{\mathbf{u}^1 - \mathbf{u}^2, p^1 - p^2\}$ would be a weak solution to the problem (3.3), (3.4) with zero $\mathbf{u}_0$, p_0, σ and $\mathbf{s}$. Hence, from estimate (3.8) we would have $\mathbf{u}^1 - \mathbf{u}^2 = 0$ and $p^1 - p^2 = 0$. Consequently, the problem (3.3), (3.4) has only one weak solution.

Finally, if $p_0 \in \widehat{L}^2(D)$ and $\sigma \in L^2(0, T; \widehat{L}^2(D))$, then taking $\{e_j\}_{j=1}^{\infty}$ as an orthogonal basis of $\widehat{L}^2(D)$ we obtain $p \in W^{1,2}(0, T; \widehat{L}^2(D))$. $\square$

Corollary 3.5. *Suppose that*

$$\mathbf{f} \in L^\infty(Q_T)^d, \quad \mathbf{u}_0 \in L^2(D)^d, \quad p_0 \in L^2(D).$$

Then the problem (3.1), (3.2) *has a unique weak solution* $\{\mathbf{u}, p\}$.

3.2 Incompressible Limit

When $\alpha = 0$, equations (3.2) formally turn into nonsteady Stokes equations for incompressible fluid:

$$\begin{gathered} \operatorname{div} \boldsymbol{v} = 0, \\ \rho_0 \boldsymbol{v}_t + \nabla q = \mu \Delta \boldsymbol{v} + \rho_0 \mathbf{f}. \end{gathered} \tag{3.11}$$

Consider the following initial–boundary conditions for the equations (3.11):

$$\boldsymbol{v}|_{\partial D} = 0, \quad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0. \tag{3.12}$$

The problem (3.11), (3.12) is well-studied, so we will only state some standard results for it.

Definition 3.6. A pair

$$\{\boldsymbol{v}, q\} \in L^2(0, T; V(D)) \times \mathcal{D}'(Q_T)$$

is called a *weak solution* to the problem (3.11), (3.12), if (3.11) holds in $\mathcal{D}'(Q_T)$ and for all $\varphi \in C_0^\infty([0, T))$ and $\mathbf{v} \in V(D)$

$$\begin{aligned} -\int_0^T \rho_0(\boldsymbol{v}, \mathbf{v})\, \varphi_t \, dt - \rho_0(\boldsymbol{v}_0, \mathbf{v})\varphi(0) = \\ = -\mu \int_0^T ((\boldsymbol{v}, \mathbf{v}))\, \varphi \, dt + \int_0^T (\rho_0 \mathbf{f}, \mathbf{v})\, \varphi \, dt \end{aligned} \tag{3.13}$$

Proposition 3.7. *Let* $\mathbf{f} \in L^2(0, T; H^{-1}(D)^d)$, $\boldsymbol{v}_0 \in J(D)$. *Then the problem* (3.11), (3.12) *has a unique weak solution such that*

$$\begin{gathered} \boldsymbol{v} \in \widetilde{W}^{1,2}(0, T; V(D)), \\ q = \partial_t Q, \quad Q \in C(0, T; \widehat{L}^2(D)). \end{gathered}$$

Moreover, if $\boldsymbol{v}_0 \in V(D)$ *and* $\mathbf{f} \in L^2(0,T;L^2(D))$*, then*

$$\boldsymbol{v}_t \in L^2(0,T;J(D)),$$
$$q \in L^2(0,T;\widehat{L}^2(D)).$$

Remark 3.8. The first part of the proposition is proved in [5]. The case $\boldsymbol{v}_0 \in V(D)$ is treated by a minor modification of the proof from [5].

Remark 3.9. Every weak solution to the problem (3.11), (3.12) satisfies the following *energy equality*:

$$\rho_0|\overline{\boldsymbol{v}}|^2|_\xi^\tau + 2\mu \int_\xi^\tau \|\boldsymbol{v}\|^2\,dt = 2\int_\xi^\tau (\rho_0\mathbf{f},\boldsymbol{v})\,dt \tag{3.14}$$

for all $\xi,\tau \in [0,T]$.

Regularity of the solutions to the problem (3.11), (3.12) has been rigorously studied by V.A. Solonnikov. Let us state a corollary of one of his results (see [10, p. 126]):

Proposition 3.10 (V.A. Solonnikov)**.** *Let* ℓ *be a nonnegative integer number. Let* $\partial D \in C^{2\ell+2}$*,* $\mathbf{f} \in W_2^{2\ell,\ell}(Q_T)^d$ *and* $\boldsymbol{v}_0 \in J(D) \cap W_2^{2\ell+2}(D)^d$*. Assume that all the necessary compatibility conditions up to order* ℓ *are satisfied. Then*

$$\boldsymbol{v} \in W^{2\ell+2,\ell+1}(Q_T)^d, \quad \nabla q \in W_2^{2\ell,\ell}(Q_T)^d.$$

Remark 3.11. It follows from Proposition 3.10 that $\nabla q \in W^{\ell,2}(0,T;L^2(\Omega))$. Applying ∇^{-1} to ∇q we get $q \in W^{\ell,2}(0,T;\widehat{L}^2(\Omega))$. Then for $\ell \geq 1$

$$q|_{t=0} = \nabla^{-1}P_G(\Delta\boldsymbol{v}_0 + \mathbf{f}(0)),$$

i.e. the pressure q has a well-defined initial value.

4 Convergence Towards the Incompressible Limit

In this section we suppose that the assumptions of Corollary 3.5 are satisfied. Let $\{\mathbf{u}_\alpha, p_\alpha\}_{0<\alpha<1}$ denote the family of the weak solutions to the problem (3.1), (3.2). Let $\{\boldsymbol{v}, q\}$ denote a weak solution to the problem (3.11), (3.12).

4.1 Convergence of the Velocity

Theorem 4.1. *If* $\alpha \to 0$*, then*

$$\mathbf{u}_\alpha \rightharpoonup \boldsymbol{v} \quad in\ L^2(0,T;H_0^1(D)^d), \tag{4.1}$$
$$\mathbf{u}_\alpha \rightharpoondown \boldsymbol{v} \quad in\ L^\infty(0,T;L^2(D)^d), \tag{4.2}$$
$$\nabla p_\alpha \rightharpoondown \nabla q \quad in\ H^{-1}(Q_T)^d, \tag{4.3}$$

where $\{\boldsymbol{v}, q\}$ *is the solution to* (3.11)*,* (3.12) *with*

$$\boldsymbol{v}_0 = P_J\mathbf{u}_0. \tag{4.4}$$

Proof. Consider the equality (3.6) with arbitrary $\mathbf{v} \in V(\Omega)$:

$$-\int_0^T \rho_0(\mathbf{u}_\alpha, \mathbf{v})\, \varphi_t\, dt - \rho_0(\mathbf{u}_0, \mathbf{v})\varphi(0) = -\mu \int_0^T ((\mathbf{u}_\alpha, \mathbf{v}))\, \varphi\, dt + \\ + \int_0^T (\alpha p_\alpha \mathbf{f}, \mathbf{v})\, \varphi\, dt + \int_0^T \langle \rho_0 \mathbf{f}, \mathbf{v}\rangle\, \varphi\, dt, \quad (4.5)$$

where $\varphi \in C_0^\infty([0;T))$. Clearly $(\mathbf{u}_0, \mathbf{v}) = (P_J\mathbf{u}_0, \mathbf{v})$. The estimate (3.8) implies that $(\mathbf{u}_\alpha)$ is bounded in $L^2(0,T;H_0^1(D)^d)$ and $(\sqrt{\alpha}p_\alpha)$ is bounded in $L^\infty(0,T;L^2(D))$. Then, by Alaoglu–Bourbaki theorem, for any sequence $\alpha_n \to 0$, $n \to \infty$, there exist $\boldsymbol{w} \in L^2(0,T;H_0^1(D)^d)$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\mathbf{u}_{\alpha_{n_k}} \rightharpoonup \boldsymbol{w} \ \text{ in } L^2(0,T;H_0^1(D)^d)$$

when $k \to \infty$. Passing to the limit in (4.5), we obtain

$$-\int_0^T \rho_0(\boldsymbol{w}, \mathbf{v})\, \varphi_t\, dt - \rho_0(P_J\mathbf{u}_0, \mathbf{v})\varphi(0) = \\ = -\mu \int_0^T ((\boldsymbol{w}, \mathbf{v}))\, \varphi\, dt + 0 + \int_0^T \langle \rho_0 \mathbf{f}, \mathbf{v}\rangle\, \varphi\, dt, \quad (4.6)$$

which also holds (by continuity) for each $\varphi \in H_0^1(0,T)$. Passing to the limit in (3.5) assuming that $r \in \widehat{L}^2(D)$ and $\varphi \in C_0^\infty(0,T)$ we get

$$\int_0^T (\operatorname{div} \boldsymbol{w}, r)\, \varphi\, dt = 0.$$

Since $\operatorname{div} \boldsymbol{w} \in L^2(0,T;\widehat{L}^2(D))$, this yields that $\boldsymbol{w} \in L^2(0,T;V(D))$.

Let us collect all the terms of (4.6) on its left-hand side and denote the resulting expression by $\Phi(\mathbf{v},\varphi)$. For any fixed $\mathbf{v} \in H_0^1(D)^d$

$$\Phi(\mathbf{v},\cdot) \in H^{-1}(0,T).$$

Integrating (4.6) by parts we see that $\Phi(\mathbf{v},\varphi) = \int_0^T \phi(\mathbf{v},t)\varphi_t\, dt$, where

$$\phi(\mathbf{v},t) = -\rho_0(\boldsymbol{w}, \mathbf{v}) - \mu \int_0^t ((\boldsymbol{w}, \mathbf{v}))\, dt + \int_0^t \langle \rho_0 \mathbf{f}, \mathbf{v}\rangle\, dt$$

Clearly $\phi(\cdot,t) \in C(0,T;H^{-1}(D)^d)$. From (4.6) and Lemma 2.12 it follows that there exists a unique $Q \in C(0,T;\widehat{L}^2(D))$ such that for $t \in [0,T]$

$$\phi(\cdot,t) = \langle \nabla Q(t), \cdot\rangle$$

Let p denote the distributional derivative $\partial_t Q$ in $\mathcal{D}'(Q_T)$. Then $\{\boldsymbol{w}, p\}$ is a weak solution to the problem (3.11), (3.12) with the initial condition (4.4). By Proposition 3.7 this solution is unique, so $\{\boldsymbol{w}, p\} = \{\boldsymbol{v}, q\}$. Hence the whole

family $\mathbf{u}_\alpha$ converges to $\boldsymbol{v}$ when $\alpha \to 0$, i.e. (4.1) holds. (Otherwise this family would have a subsequence which wouldn't converge to $\boldsymbol{v}$ weakly. But such a subsequence would be bounded and applying Alaoglu–Bourbaki theorem and the arguments above we would come to a conclusion that $\mathbf{u}_\alpha$ converges to $\boldsymbol{v}$, which is a contradiction.)

As a consequence of estimate (3.8) and Alaoglu–Bourbaki theorem, we can pass, if necessary, to another subsequence of (n_k), so that $\mathbf{u}_{\alpha_{n_k}} \rightharpoonup \mathbf{w}$ in $L^\infty(0,T;L^2(D)^d)$ for some $\mathbf{w}$. Hence $\forall \varphi \in C_0^\infty(0,T)$ and $\forall \mathbf{r} \in L^2(D)^d$

$$\int_0^T (\mathbf{u}_{\alpha_{n_k}}, \mathbf{r})\varphi dt \to \int_0^T (\mathbf{w}, \mathbf{r})\varphi dt, \quad k \to \infty.$$

From this equation and (4.1) we conclude that $\mathbf{w} = \boldsymbol{v}$ for a.e. $t \in [0,T]$. Thus $\mathbf{u}_{\alpha_{n_k}} \rightharpoonup \boldsymbol{v}$ in $L^\infty(0,T;L^2(D)^d)$. Since the whole family $\mathbf{u}_\alpha$ converges to $\boldsymbol{v}$ when $\alpha \to 0$, reasoning as above we conclude that (4.2) holds.

Finally, (4.3) follows from (4.1) and the passage to the limit in (3.6) when $\alpha \to 0$. □

Theorem 4.1 is in good agreement with the results proved in [4, 1]. But the topology of the convergence (4.1), (4.3) can be strengthened using the approach coming from the artificial compressibility method [5]:

Theorem 4.2. *Suppose that $\boldsymbol{v}_0 = P_J \mathbf{u}_0$. When $\alpha \to 0$,*

$$\mathbf{u}_\alpha \to \boldsymbol{v} \quad \text{in } L^2(0,T;H_0^1(D)^d), \tag{4.7}$$

$$\mathbf{u}_\alpha \to \boldsymbol{v} \quad \text{in } L^\infty(0,T;L^2(D)^d), \tag{4.8}$$

$$\nabla p_\alpha \to \nabla q \quad \text{in } H^{-1}(Q_T)^d, \tag{4.9}$$

if and only if $\mathbf{u}_0 \in J(D)$.

Proof. By Remark 2.4 and Theorem 4.1 $\forall \mathbf{v} \in V(D)$

$$\begin{aligned}
&\rho_0(\overline{\mathbf{u}}_\alpha(t) - \mathbf{u}_0, \mathbf{v}) + 0 = \\
&\quad = -\mu \int_0^t ((\mathbf{u}_\alpha, \mathbf{v}))dt - \eta \int_0^t (\operatorname{div} \mathbf{u}_\alpha)(\operatorname{div} \mathbf{v})dt + \int_0^t (\rho_0 + \alpha p)(\mathbf{f}, \mathbf{v})dt \to \\
&\qquad \to -\mu \int_0^t ((\boldsymbol{v}, \mathbf{v}))dt - 0 + \int_0^t \rho_0(\mathbf{f}, \mathbf{v})dt = \rho_0(\overline{\boldsymbol{v}}(t) - \boldsymbol{v}_0, \mathbf{v}).
\end{aligned}$$

By estimate (3.8) $|\overline{\mathbf{u}}_\alpha(t)|$ is bounded for $0 < \alpha < 1$. Then, since $V(D)$ is dense in $J(D)$, by Banach–Steinhaus theorem $\forall \mathbf{v} \in J(D)$

$$(\overline{\mathbf{u}}_\alpha(t), \mathbf{v}) \to (\overline{\boldsymbol{v}}(t), \mathbf{v}). \tag{4.10}$$

Now consider a family of real numbers

$$\begin{aligned}
X_\alpha = \rho_0 |\overline{\mathbf{u}}_\alpha(T) - \overline{\boldsymbol{v}}(T)|^2 + \frac{\alpha}{\rho_0} |\overline{p}_\alpha(T)|^2 + \\
+ 2\mu \int_0^T \|\mathbf{u}_\alpha - \boldsymbol{v}\|^2 \, dt + 2\eta \int_0^T |\operatorname{div}(\mathbf{u}_\alpha - \boldsymbol{v})|^2 \, dt.
\end{aligned}$$

Expanding the parentheses we write

$$X_\alpha = X^{(1)} + X_\alpha^{(2)} + X_\alpha^{(3)},$$

where

$$X^{(1)} = \rho_0 |\overline{\boldsymbol{v}}(T)|^2 + 2\mu \int_0^T \|\boldsymbol{v}\|^2 \, dt,$$

$$X_\alpha^{(2)} = -2\rho_0 (\overline{\mathbf{u}}_\alpha(T), \overline{\boldsymbol{v}}(T)) - 4\mu \int_0^T ((\mathbf{u}_\alpha, \boldsymbol{v})) \, dt,$$

$$X_\alpha^{(3)} = \rho_0 |\overline{\mathbf{u}}_\alpha(T)|^2 + \frac{\alpha}{\rho_0} |\overline{p}_\alpha(T)|^2 + 2\mu \int_0^T \|\mathbf{u}_\alpha\|^2 \, dt + 2\eta \int_0^T |\operatorname{div} \mathbf{u}_\alpha|^2 \, dt.$$

By energy equality (3.14)

$$X^{(1)} = \rho_0 |\boldsymbol{v}_0|^2 + \int_0^T (\rho_0 \mathbf{f}, \boldsymbol{v}) \, dt.$$

From (4.10) and (4.1)

$$X_\alpha^{(2)} \to -2\rho_0 (\overline{\boldsymbol{v}}(T), \overline{\boldsymbol{v}}(T)) - 4\mu \int_0^T ((\boldsymbol{v}, \boldsymbol{v})) \, dt = -2X^{(1)}.$$

Now we pass to the limit in energy equality (3.7) using estimate (3.8):

$$X_\alpha^{(3)} = \rho_0 |\mathbf{u}_0|^2 + \frac{\alpha}{\rho_0} |p_0|^2 + \int_0^T ((\rho_0 + \alpha p)\mathbf{f}, \mathbf{u}_\alpha) \, dt \to$$
$$\to \rho_0 |\mathbf{u}_0|^2 + 0 + \int_0^T (\rho_0 \mathbf{f}, \boldsymbol{v}) \, dt = X^{(1)} + \rho_0 (|\mathbf{u}_0|^2 - |\boldsymbol{v}_0|^2).$$

Hence we have proved that $X_\alpha \to \rho_0(|\mathbf{u}_0|^2 - |\boldsymbol{v}_0|^2)$ when $\alpha \to 0$. Then (4.7) and (4.8) hold if and only if $|\mathbf{u}_0| = |\boldsymbol{v}_0| = |P_J \mathbf{u}_0|$, i.e. $\mathbf{u}_0 \in J(D)$. And the passage to the limit in (3.6) implies (4.9). $\square$

4.2 Convergence of the Pressure

The important difference between the equations of compressible fluid and the equations of incompressible fluid is that only the former enjoy the mass conservation property. Namely, let

$$M(t) = \int_D \rho \, dx = \int_D (\rho_0 + \alpha p) \, dx.$$

Then (3.1) implies

$$\frac{dM}{dt} = 0.$$

Consequently $\int_D p(t) \, dx$ doesn't depend on t. In contrast, this is not true for the incompressible fluid, since q can be shifted by arbitrary function of t.

However, for arbitrary $A \in \mathbb{R}$ the pressure q can be shifted so that $\int_D q(t)\,dx = A$ for a.e. $t \in [0,T]$. (Let $C(t) = (\int q(t)\,dx - A)/\int_D dx$, then the substitution $q(t) := q(t) - C(t)$ yields the desired result.)

Theorem 4.3. *Suppose that* $\boldsymbol{v}_0 = \mathbf{u}_0 \in J(D)$ *and*

$$\boldsymbol{v} \in \widetilde{W}^{1,2}(0,T;H_0^1(D)^d), \qquad q \in W^{1,2}(0,T;L^2(D)),$$

$$\int_D q(t)\,dx = \int_D p_0\,dx \quad \textit{for a.e. } t \in [0,T] \tag{4.11}$$

Then

$$p_\alpha \rightharpoonup q \ \ \textit{in } L^\infty(0,T;L^2(D)), \quad \alpha \to 0.$$

Proof. Assumptions of the theorem imply that the solution $\{\boldsymbol{v}, q\}$ to the "incompressible" problem (3.11), (3.12) belongs to the same functional space as the solution to the "compressible" problem (3.1), (3.2). Then the difference of the solutions $\{\mathbf{U}_\alpha, P_\alpha\} \equiv \{\mathbf{u}_\alpha - \boldsymbol{v}, p_\alpha - q\}$ is a weak solution to the problem (3.3), (3.4) with initial conditions

$$\mathbf{U}_\alpha|_{t=0} = 0, \qquad P_\alpha|_{t=0} = P_0 \equiv p_0 - q(0)$$

and with non-homogeneous terms given by

$$\sigma = -\alpha q_t, \quad \mathbf{s} = \alpha q \mathbf{f}.$$

Assumption (4.11) implies $q_t \in L^2(0,T;\widehat{L}^2(D))$. Then (3.8) implies

$$\|\mathbf{U}_\alpha\|_{L^2(0,T;H_0^1(D)^d)} + \|\mathbf{U}_\alpha\|_{L^\infty(0,T;L^2(D)^d)} \le C\sqrt{\alpha}E,$$
$$\|P_\alpha\|_{L^\infty(0,T;\widehat{L}^2(D))} \le CE,$$
$$E = \|P_0\|_{\widehat{L}^2(D)} + \|q_t\|_{L^2(0,T;L^2(D))} + \sqrt{\alpha}\|\mathbf{f}\|_\infty \|q\|_{L^2(0,T;L^2(D))}$$

thus we have proved the convergence of the velocity (once again).

Uniform boundedness of the pressure P_α, by Alaoglu–Bourbaki theorem, implies that for every sequence $\alpha_n \to 0$, $n \to \infty$, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$P_{\alpha_{n_k}} \rightharpoonup P_* \quad \text{in } L^\infty(0,T;L^2(D)), \quad k \to \infty,$$

where $P_* \in L^\infty(0,T;\widehat{L}^2(D))$. Passing to the limit in the equation (3.6) (as in the proof of Theorem 4.1) we show that ∇P_* is zero. Then $P_* = 0$ and the whole family P_α converges to zero. □

Remark 4.4. The estimate (3.8) not only yields the *fact* of convergence, but also gives the *rate* of the convergence. In other words, (3.8) allows to estimate the *error of approximation* of a compressible fluid by the incompressible one.

It would be desirable to have *strong* convergence of the pressure. But in general case such convergence is impossible, since (due to Proposition 2.1)

$$\|p_\alpha - q\|_{L^\infty(0,T;L^2(D))} \ge \|p_0 - \overline{q}(0)\|_{L^2(D)}.$$

Theorem 4.5. *Suppose the assumptions of Theorem 4.3 are satisfied and the domain D is star-shaped with respect to some ball. If, in addition,*

$$q \in W^{2,2}(0,T;L^2(D)) \qquad \text{and} \qquad p_0 = \overline{q}(0),$$

then

$$p_\alpha \to q \quad \text{in} \quad L^\infty(0,T;L^2(D)), \qquad \alpha \to 0.$$

Proof. Let $\mathbf{w} = -\mathcal{B}(q_t)/\rho_0$. By Proposition 2.13

$$\mathbf{w} \in W^{1,2}(0,T;H_0^1(D)^d)$$

and

$$\|\mathbf{w}\|_{W^{1,2}(0,T;H_0^1(D)^d)} \le C\|q_t\|_{W^{1,2}(0,T;L^2(D))}.$$

Then $\{\mathbf{U}_\alpha, P_\alpha\} \equiv \{\mathbf{u}_\alpha - \boldsymbol{v} - \alpha\mathbf{w}, p_\alpha - q\}$ is a weak solution to the problem (3.3), (3.4) with initial conditions

$$\mathbf{U}_\alpha|_{t=0} = \mathbf{U}_0 \equiv -\alpha w(0), \qquad P_\alpha|_{t=0} = P_0 \equiv p_0 - \overline{q}(0) = 0$$

and with non-homogeneous terms given by

$$\sigma = 0, \quad \mathbf{s} = \alpha(-\rho_0\mathbf{w}_t + \mu\Delta\mathbf{w} + \eta\nabla\operatorname{div}\mathbf{w}) + \alpha q\mathbf{f}.$$

Clearly

$$\|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} \le \alpha C C_1,$$

where

$$C_1 = \|\mathbf{w}\|_{W^{1,2}(0,T;H_0^1(D)^d)} + \|\mathbf{f}\|_\infty \|q\|_{L^2(0,T;L^2(D))}.$$

Assumption (4.11) implies $q_t \in L^2(0,T;\widehat{L}^2(D))$. Then, from (3.8)

$$\|\mathbf{U}_\alpha\|_{L^2(0,T;H_0^1(D)^d)} \le C\sqrt{\alpha}E,$$
$$\|P_\alpha\|_{L^\infty(0,T;\widehat{L}^2(D))} \le CE,$$

where

$$E \equiv (\|\mathbf{U}_0\|_{L^2(D)^d} + \sqrt{\alpha}\|P_0\|_{L^2(D)} + \\ + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)})/\sqrt{\alpha} \le CC_1\sqrt{\alpha},$$

which yields the desired convergence when $\alpha \to 0$. □

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